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PROOF OF THE THEOREM THAT EVERY EQUATION HAS A ROOT.

BY JOHN MACNIE, A. M., NEW YORK.

LET $f(x)$ be a rational integral algebraic function of the nth degree, where n is an even number; if $f(x)$ does not vanish for any real value of x, it will for some value of x of the form $A+B\sqrt{-1}$, where B is not zero.

In $f(x)$ putting $y+z\sqrt[n]{-1}$ for x, we have:

$$f(y+z\sqrt[n]{-1})=f(y)+f_1(y)z\sqrt[n]{-1}+\tfrac{1}{2}f_2(y)z^2(\sqrt[n]{-1})^2+\dots$$
$$+\frac{1}{(n-1)!}f_{n-1}(y)z^{n-1}(\sqrt[n]{-1})^{n-1}-z^n. \quad [1]$$

Now $\sqrt[n]{-1}$, as is proved independently of this proposition, must be of the form $A+B\sqrt{-1}$, equal, say, $a_1+b_1\sqrt{-1}$; its powers mast also be of the same form, say, $a_2+b_2\sqrt{-1}$, $a_3+b_3\sqrt{-1}$, ... $a_{n-1}+b_{n-1}\sqrt{(-1)-1}$, where a_1, a_2, ... b_1, b_2, ... are positive or negative, but not both zero in the same power of $\sqrt[n]{-1}$. Substituting these values for $\sqrt[n]{-1}$ and its powers in [1], we have:

$$f(y)+z\sqrt[n]{-1}=f(y)+f_1(y)(a_1+b_1\sqrt{-1})z+\tfrac{1}{2}f_2(y)(a_2+b_2\sqrt{-1})z^2+$$
$$\dots\frac{1}{(n-1)!}f_{n-1}(y)(a_{n-1}+b_{n-1}\sqrt{-1})z^{n-1}-z^n$$
$$=f(y)+\frac{a_1}{1}f_1(y)z+\frac{a_2}{2}f_2(y)z^2+\dots\frac{a_{n-1}}{(n-1)!}f_{n-1}(y)z^{n-1}-z^n$$
$$+\left[\frac{b_1}{1}f_1(y)+\frac{b_2}{2}f_2(y)z+\frac{b_3}{3!}f_3(y)z^2+\dots\frac{b_{n-1}}{(n-1)!}f_{n-1}(y)z^{n-2}\right]z\sqrt{-1}$$
$$=P+Qz\sqrt{-1};\text{ or}$$

$$-P=z^n-\frac{a_{n-1}}{(n-1)!}f_{n-1}(y)z^{n-1}-\dots-\frac{a^2}{2}f_2(y)z^2-\frac{a_1}{1}f_1(y)z-f(y). \quad [2]$$

$$Q=\frac{b_{n-1}}{(n-1)!}f_{n-1}(y)z^{n-2}+\frac{b_{n-2}}{(n-2)!}f_{n-2}(y)z^{n-3}+\dots\frac{b_2}{2}f_2(y)z+\frac{b_1}{1}f_1(y). \quad [3]$$

What we have to prove is that there must be some values of y and z for which [2] and [3] will vanish simultaneously.

$-P$, being a function of even degree with its final term negative,—since, by hypothesis, $f(y)$ cannot be negative for any real value of y,—must vanish for some real positive value of z, whatever real value, positive or negative, is given to y: but these values cannot include either infinity or zero, since neither the first nor the last ceffiicient can be zero. (See Note 1.) These positive values of z for which $-P$ can be made to vanish, vary continuously with y. For suppose $-P=0$ when $y=c$, $z=t$; then

$$-P(c,t)=t^n-\frac{a_{n-1}}{(n-1)!}f_{n-1}(c)t^{n-1}-\dots\frac{a_1}{1}f_1(c)t-f(c)=0.$$

When y becomes $c+h$ we have

$$-P(c+h,t)=-P(c,t)-h\left[\frac{a_{n-1}}{(n-1)!}\Big(f_{n-2}(c)+\&c.\Big)t^{n-1}\ldots\ldots \right.$$
$$\left. +\frac{a_1}{1}\Big(f_2(c)+\&c.\Big)t+\Big(f_1(c)+\&c.\Big)\right]$$

As the difference between $-P(c,t)$ and $-P(c+h,t)$ may be made as small as we please by taking h small enough, $-P(c+h,z)$ will vanish for a value of z differing from t by an indefinitely small quantity, if h is indefinitely small. For, regarding $c+h$ as constant and z as variable, we know that, corresponding to an indefinitely small change in the value of z there will be an indefinitely small change in the value of the function $-P(c+h,z)$.

Q, again, is a function of even degree in z with its final term negative for every value of y that makes $f_1(y)$ negative, if b_1 is positive; or for every value of y that makes $f_1(y)$ positive, if b_1 is negative. But $f_1(y)$, being a function of odd degree in y, can become zero for at least one real value of y, and positive or negative for an indefinite number of such values. When we assign to y that value for which $f_1(y)$ vanishes, then Q will vanish for $z=0$; and when we assign to y that value for which $f_{n-1}(y)$–a function of the first degree–vanishes, then Q will vanish for $z=\infty$, (See NOTE 1.) that is, by assigning appropriate values to y, Q may be made to vanish for positive values of z indefinitely small or great. That the values of z between these limits for which Q may be made to vanish vary continuously with y may be shown as in regard to $-P$.

Now when we assign to y in both $-P$ and Q that value (c_1) that makes $f_1(y)$ vanish, the value of z (t_1) for which $-P$ will vanish is greater than that (i. e. zero) for which Q will vanish; while if we assign to y, in both $-P$ and Q, that value (c_2) for which $f_{n-1}(y)$ vanishes, the value of z (t_2) for which $-P$ vanishes will be less than that (i. e. infinity) for which Q vanishes. If, therefore, we conceive y, in both $-P$ and Q, to pass through all values from c_1 to c_2, y must reach some intermediate value (c_3) at which the value of z (t_3), causing $-P$ to vanish, ceases to be greater than that which causes Q to vanish; that is, $\mp P=0$ and $Qz\sqrt{-1}=0$ simultaneously for $y=c_3$, $z=t_3$, or $f(x)=0$ when $x=c_3+t_3(a_1\pm b_1\sqrt{-1})=(c_3+a_1t_3)$ $\pm b_1t_3\sqrt{-1}$, a quantity of the form $A+B\sqrt{-1}$, in which A and B are real finite quantities since a_1, b_1, c_3, t_3, are real and finite.

NOTE 1. The theorem, that the equation having for coefficients those of a proposed equation taken in reverse order has for roots the reciprocals of the roots, if any, of the proposed eqation, is independent of the property of equations here discussed. From this, in connection with the obvious principle that for $f(z)$ to vanish for $z=0$ it is necessary and sufficient that the absolute term be zero, follows the principle here employed, that for $f(z)$ to vanish for $z = \infty$ it is necessary and sufficient that the coefficient of its highest power be evanescent.

NOTE 2. We have found that a pair of roots of $f(x) = 0$ may be expressed by $y_p+z_p(\alpha_p\pm\beta_p\sqrt{-1})$, where we shall suppose $\alpha_p\pm\beta_p\sqrt{-1}$ to denote the values of $\sqrt[n]{-1}$ given by the formula

$$\cos\frac{(2\lambda+1)\pi}{n} \pm \sin\frac{(2\lambda+1)\pi}{n}\sqrt{-1},$$

where $\lambda = p$, p being any integer from 0 to n inclusive. A pair is also expressed by $y_q + z_q(\alpha_q \pm \beta_q\sqrt{-1})$, where $\alpha_q \pm \beta_q\sqrt{-1}$ denotes the values of $\sqrt[n]{-1}$ given by the above trigonometrical formula when $\lambda = q$, q being any integer, except p, from 0 to n. The question arises: does $y_q + z_q(\alpha_q \pm \beta_q\sqrt{-1})$ necessarily represent a different pair of roots from that represented by $y_p+z_p(\alpha_p\pm\beta_p\sqrt{-1})$, by different here meaning not equal. This conclusion seems necessary; since, otherwise, there would be more than one expression for some pairs of roots, and none at all for others, there being but $\frac{1}{2}n$ pairs of values of $\sqrt[n]{-1}$. If we can assume this result we are led to an interesting generalization respecting the roots of all equations of, at least, even degree, which could then be expressed by the formula

$$x = y + z\left[\cos\frac{(2\lambda+1)\pi}{n} \pm \sin\frac{(2\lambda+1)\pi}{n}\sqrt{-1}\right].$$

In this when $y = 0$, $z = \sqrt[n]{A}$, we have the well known formula for the roots of $x^n + A = 0$, when n is even; for $z = 0$, the formula represents real roots; if $z_p\beta_p = z_q\beta_q$ and $y_p+z_p\alpha_p = y_q + z_q\alpha_q$, there are two pairs of equal imaginary roots, or one pair of equal real roots if $z = 0$. The subject seems worthy of further investigation.

A COLLECTION OF PROOFS OF THE RELATION $r'+r''+r'''-r = 4R$.

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IF R is the radius of the circle circumscribed about a plane triangle, r the radius of the inscribed circle, and if r', r'', r''' be the radii of the escribed circles, then is the above relation true. The earliest appearance of this relation in print that I have seen is in J. H. Van Swinden's Elemente der geometrie von C. F. A. Jacobi: Jena, 1834; though the relation was doubtless found earlier than this. The collection here made is not the result of a special effort to get all the proofs known of it but rather to put together such as have been met with incidentally in mathematical reading.